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# AN EXPLICIT CONSTRUCTION OF SIMPLE-MINDED SYSTEMS OVER SELF-INJECTIVE NAKAYAMA ALGEBRAS

BY

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**Abstract.** Recently, we obtained a new characterization for an orthogonal system to be a simple-minded system in the stable module category of any representation-finite self-injective algebra. In this paper, we apply this result to give an explicit construction of simple-minded systems over self-injective Nakayama algebras.

**1. Introduction.** In her famous work on classification of representation-finite self-injective algebras $A$ over an algebraically closed field $k$, Riedtmann defined the notion of (combinatorial) configurations in the stable Auslander–Reiten quiver ${}_s\Gamma_A$ of $A$. It turns out that the configurations of ${}_s\Gamma_A$ precisely correspond to simple-minded systems (sms for short) of the stable module category $A$-$\underline{\mathrm{mod}}$ (see [6]). In Riedtmann and her collaborators' work ([13], [14], [15], [4]), the classification of sms's over any representation-finite self-injective algebra has been theoretically completed. In particular, if $A$ is the self-injective Nakayama algebra with $n$ simple modules and Loewy length $\ell + 1$, then the sms's of $A$-$\underline{\mathrm{mod}}$ are classified by $\tau^n$-stable Brauer relations of order $\ell$. Recently, Chan [5] gave a new classification of sms's over self-injective Nakayama algebras in terms of two-term tilting complexes.

Both Riedtmann's and Chan's classifications are implicit and it is not easy to write down the sms's explicitly from these classifications. In the present paper, we give an explicit construction of sms's over self-injective Nakayama algebras. Our construction depends on a new characterization of sms's over representation-finite self-injective algebras in [7] and a description (see Proposition 3.3 below) of the orthogonality condition in the stable module category over any self-injective Nakayama algebra.

We now briefly state our main result. Let $A$ be the self-injective Nakayama algebra with $n$ simple modules and Loewy length $\ell + 1$, and let $\mathcal{P}$ be the set

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of non-crossing partitions of $\underline{e} := \{1, \ldots, e\}$, where $e$ is the greatest common divisor of $n$ and $\ell$. For each pair $(p, k)$ where $p \in \mathcal{P}$ and $k \in \underline{e}$, we construct two explicit families $\mathcal{L}'_{p,k}$ and $\mathcal{S}'_{p,k}$ of $A$-modules, and we prove that these families consist a complete set of sms's over $A$ (see Theorems 4.10 and 4.16). The virtue of our construction is that one can directly write down the modules in the sms's from non-crossing partitions.

This paper is organized as follows. In Section 2, we recall some notions and facts on sms's and on self-injective Nakayama algebras. In Section 3, we introduce the arc of indecomposable module over any symmetric Nakayama algebra and use it to describe orthogonality in the corresponding stable module category. In Section 4, we recall non-crossing partitions and give an explicit construction of sms's over self-injective Nakayama algebras. In the last section, we study the behavior of our construction under the (co)syzygy functor.

**2. Preliminaries.** Throughout this paper all algebras will be finite-dimensional algebras over an algebraically closed field $k$. For the details on representations of algebras and quivers we refer to [2]. For an algebra $A$, we denote by $A$-mod the category of finite-dimensional (left) $A$-modules. For any $A$-module $M$, we denote by $\mathrm{soc}(M)$ and $\mathrm{rad}(M)$ the socle and the radical of $M$, respectively. We shall use the following notations: $\mathrm{rad}^0(M) := M$, $\mathrm{rad}^{k+1}(M) := \mathrm{rad}(\mathrm{rad}^k(M))$ for $k \in \mathbb{N}$ and $\mathrm{top}(M) := M/\mathrm{rad}(M)$. Recall that the *stable module category* $A$-$\underline{\mathrm{mod}}$ of $A$-mod has the same objects as $A$-mod but the morphism space between two objects $M$ and $N$ is the quotient space $\underline{\mathrm{Hom}}_A(M, N) := \mathrm{Hom}_A(M, N)/\mathcal{P}(M, N)$, where $\mathcal{P}(M, N)$ is the subspace of $\mathrm{Hom}_A(M, N)$ consisting of those homomorphisms from $M$ to $N$ which factor through a projective $A$-module.

The notion of simple-minded system (sms for short) was introduced by Koenig and Liu [9] in the stable module category $A$-$\underline{\mathrm{mod}}$ of any finite-dimensional algebra $A$. It was shown in [9] that when $A$ is representation-finite self-injective, the sms's in $A$-$\underline{\mathrm{mod}}$ can be defined as follows.

Definition 2.1 ([9, Theorem 5.6]). Let $A$ be a representation-finite self-injective algebra. A family of objects $\mathcal{S}$ in $A$-$\underline{\mathrm{mod}}$ is an *sms* if the following conditions are satisfied:

(1) For any two objects $S, T$ in $\mathcal{S}$,

$$\underline{\mathrm{Hom}}_A(S, T) \cong \begin{cases} 0 & (S \neq T), \\ k & (S = T). \end{cases}$$

(2) For any indecomposable non-projective $A$-module $X$, there exists $S$ in $\mathcal{S}$ such that $\underline{\mathrm{Hom}}_A(X, S) \neq 0$.

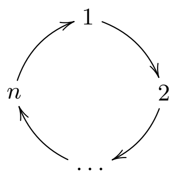
Recently, we obtained in [7] a new characterization of sms's over representation-finite self-injective algebras. To state the characterization, we first introduce the following definition.

Definition 2.2 (cf. [7, Definition 2.1]). Let $A$ be a self-injective algebra and $M$ an indecomposable $A$-module. $M$ is a *stable brick* if $\underline{\mathrm{Hom}}_A(M,M)\cong k$. A set $S$ of stable bricks in $A$-$\underline{\mathrm{mod}}$ is an *orthogonal system* if $\underline{\mathrm{Hom}}_A(M,N)=0$ for all distinct stable bricks $M$, $N$ in $S$.

Theorem 2.3 ([7, Theorem 3.1]). *Let $A$ be a representation-finite self-injective algebra and $\mathcal{S}$ a family of objects in $A$-$\underline{\mathrm{mod}}$. Then $\mathcal{S}$ is an sms if and only if $\mathcal{S}$ satisfies the following three conditions:*

(1) *$\mathcal{S}$ is an orthogonal system in $A$-$\underline{\mathrm{mod}}$.*
(2) *The cardinality of $\mathcal{S}$ is equal to the number of non-isomorphic non-projective simple $A$-modules.*
(3) *$\mathcal{S}$ is Nakayama-stable, that is, the Nakayama functor $\nu$ permutes the objects of $\mathcal{S}$.*

Now we specialize our discussion to self-injective Nakayama algebras. We denote by $A_n^\ell$ the self-injective Nakayama algebra with $n$ simples and Loewy length $\ell+1$, where $n,\ell$ are positive integers. More precisely, $A_n^\ell = kQ/I$ is given by the following quiver $Q$:




with admissible ideal $I=\mathrm{rad}^{\ell+1}(kQ)$. It is known that $A_n^\ell$ is representation-finite [2, V.3, Theorem 3.5].

Let $A=A_n^\ell$ be a self-injective Nakayama algebra defined as above. As usual, we denote by $D$, $\nu$, $\Omega$, and $\tau = D\,\mathrm{Tr}$ the $k$-dual functor, the Nakayama functor, the syzygy functor and the Auslander–Reiten translate of $A$, respectively. Let $S_1,\dots,S_n$ be the simple $A$-modules corresponding to the vertices $1,\dots,n$ of the quiver $Q$. For any indecomposable $A$-module $M$, the Loewy length of $M$, denoted by $\ell(M)$, is the number of composition factors in any composition series of $M$. Notice that any indecomposable $A$-module $M$ is uniserial and completely determined up to isomorphism by $\mathrm{top}(M)$, $\mathrm{soc}(M)$ and $\ell(M)$. We write $M=M^i_{j,k}$ to indicate that $\mathrm{top}(M)$ is isomorphic to $S_i$, $\mathrm{soc}(M)$ is isomorphic to $S_j$, and the multiplicity of $S_i$ in $M$ (that is, the number of composition factors of $M$ which are isomorphic to $S_i$) is $k+1$. Moreover, the dimension of $M^i_{j,k}$ as vector space is $nk+[j-i)+1$, where $[j-i)$ is the smallest non-negative integer with $[j-i)=(j-i) \bmod n$. If $i<j$, then

the dimension vector of $M^i_{j,k}$ is $(k, k, \dots, k, k+1, k+1, \dots, k+1, k, k, \dots, k)$, where $k+1$ appears from position $i$ to position $j$. If $i > j$, then the dimension vector of $M^i_{j,k}$ is $(k+1, k+1, \dots, k+1, k, \dots, k, k+1, k+1, \dots, k+1)$, where $k+1$ appears from position 1 to position $j$ and from position $i$ to position $n$. If $i = j$, then the dimension vector of $M^i_{j,k}$ is $(k, k, \dots, k, k+1, k, k, \dots, k)$, where $k+1$ appears at position $i$. In the following, we will freely use the above notation or a Loewy diagram as in Example 4.3 to specify an indecomposable $A^\ell_n$-module.

The Nakayama functor $\nu$ of $A$ is important to the present paper and we give a description for it in the following two lemmas.

LEMMA 2.4. *Let $A^\ell_n = kQ/I$ be a self-injective Nakayama algebra. If $M$ is an indecomposable non-projective $A^\ell_n$-module, then $\nu(M) \cong \tau^{-\ell}(M)$.*

*Proof.* We can easily verify this result for simple modules and then extend it to all indecomposable non-projective modules since $\nu$ is a self-equivalence over $A$-mod. ∎

LEMMA 2.5. *Let $M$ be an indecomposable non-projective $A^\ell_n$-module. We denote by $O_\nu(M)$ the $\nu$-orbit of $M$. Then the number of objects in $O_\nu(M)$ is $n/e$ and $O_\nu(M) = \{M, \tau^{-e}(M), \dots, \tau^{-n+e}(M)\}$, where $e = \gcd(n, \ell)$.*

*Proof.* By Lemma 2.4, we have $O_\nu(M) = \{M, \tau^{-\ell}(M), \dots, \tau^{-(k-1)\ell}(M)\}$, where $k$ is the minimum positive integer such that $n$ divides $k\ell$. Since $n/e$ and $\ell/e$ are coprime, we have $k = n/e$. Thus, the number of objects in $O_\nu(M)$ is $n/e$ and $O_\nu(M) = \{M, \tau^{-e}(M), \dots, \tau^{-n+e}(M)\}$. ∎

In the rest of this section, we prove several elementary results on homomorphism spaces in the stable category of a self-injective Nakayama algebra. For $f \in \operatorname{Hom}_A(M, N)$, we will denote its image in $\underline{\operatorname{Hom}}_A(M, N)$ by $\underline{f}$.

LEMMA 2.6. *Let $A = A^\ell_n$ be a self-injective Nakayama algebra, and let $M, N$ be indecomposable non-projective $A$-modules. Suppose that there exists a non-zero morphism $f \in \operatorname{Hom}_A(M, N)$ satisfying $\operatorname{Im} f = \operatorname{rad}^t(N)$. Then $\underline{f} = 0$ if and only if $\ell(M) + t \geq \ell + 1$. In particular, if $i$ (respectively $j$) is the multiplicity of $\operatorname{top}(N)$ in $N/\operatorname{rad}^t(N)$ (respectively $M$), then $\underline{f} = 0$ implies $i + j \geq [\ell/n] + 1$, where $[\ell/n]$ is the maximum integer no more than $\ell/n$.*

*Proof.* "⇒" Since $\underline{f} = 0$, we have the following commutative diagram in $A$-mod:

$$\begin{tikzcd} M \arrow[rr, "f"] \arrow[dr, "g"'] & & N \\ & P_N \arrow[ur, "\pi"'] & \end{tikzcd}$$

where $\pi$ is the projective cover of $N$. Then $\operatorname{Im} g = \operatorname{rad}^t(P_N)$ and $\ell(M) \geq \ell(\operatorname{rad}^t(P_N)) = \ell(P_N) - t = \ell + 1 - t$, that is, $\ell(M) + t \geq \ell + 1$.

"$\Leftarrow$" Suppose that $\ell(M)+t \geq \ell+1$ and let $\pi\colon P_N \to N$ be the projective cover of $N$. Then we can define a morphism $g$ from $M$ to $P_N$ satisfying $\operatorname{Im} g = \operatorname{rad}^t(P_N)$ and $f = \pi g$, that is, $f$ factors through a projective module. ∎

Remark 2.7. Notice that $i+j \geq [\ell/n]+1$ is not a necessary and sufficient condition for $\underline{f} = 0$ in general. However, if $A_n^\ell$ is a symmetric Nakayama algebra (that is, there exists an integer $d$ such that $\ell = dn$), then the condition $i+j \geq d+1$ is a necessary and sufficient condition for $\underline{f} = 0$.

Lemma 2.8. *Let $M$ and $N$ be indecomposable non-projective $A_n^\ell$-modules. Let $f \in \operatorname{Hom}_{A_n^\ell}(M,N)$ be a non-zero homomorphism such that $\operatorname{Im} f = \operatorname{rad}^t(N)$, where $t$ is an integer such that there is no epimorphism from $M$ to $\operatorname{rad}^s(N)$ for $s < t$. Then $\underline{f} = 0$ if and only if $\underline{\operatorname{Hom}}_{A_n^\ell}(M,N) = 0$.*

*Proof.* "$\Leftarrow$" When $\underline{\operatorname{Hom}}_{A_n^\ell}(M,N) = 0$, it is clear that $\underline{f} = 0$.

"$\Rightarrow$" If $\underline{f} = 0$, then $\ell(M)+t \geq \ell+1$ by Lemma 2.6. For any morphism $g$ in $\operatorname{Hom}_{A_n^\ell}(M,N)$, since there is no epimorphism from $M$ to $\operatorname{rad}^s(N)$ $(s<t)$, we have $\operatorname{Im} g = \operatorname{rad}^s(N)$ for some integer $s$, where $s \geq t$. Therefore $\ell(M)+s \geq \ell+1$, and again by Lemma 2.6, $\underline{g} = 0$. This shows $\underline{\operatorname{Hom}}_{A_n^\ell}(M,N) = 0$. ∎

Lemma 2.9. *Let $A_n^\ell$ be a self-injective Nakayama algebra, and $M$ and $N$ indecomposable non-projective $A_n^\ell$-modules. If $\underline{\operatorname{Hom}}_{A_n^\ell}(M,N) = 0$ and $\underline{\operatorname{Hom}}_{A_n^\ell}(N,M) = 0$, then $\operatorname{top}(M) \ncong \operatorname{top}(N)$ and $\operatorname{soc}(M) \ncong \operatorname{soc}(N)$.*

*Proof.* If $\operatorname{top}(M) \cong \operatorname{top}(N)$ (respectively $\operatorname{soc}(M) \cong \operatorname{soc}(N)$), then $M$ is a quotient module (respectively submodule) of $N$ or $N$ is a quotient module (respectively submodule) of $M$, which contradicts the assumption. ∎

We now describe when the $\nu$-orbit $O_\nu(M)$ of an indecomposable non-projective $A_n^\ell$-module $M$ forms an orthogonal system in $A_n^\ell$-$\underline{\mathrm{mod}}$.

Proposition 2.10. *Let $A = A_n^\ell$ be a self-injective Nakayama algebra and $M$ an indecomposable non-projective $A$-module. Then the $\nu$-orbit $O_\nu(M)$ of $M$ is an orthogonal system in $A_n^\ell$-$\underline{\mathrm{mod}}$ if and only if $\ell(M) \leq e$ or $\ell+1-e \leq \ell(M) \leq \ell$, where $e = \gcd(n,\ell)$.*

*Proof.* "$\Leftarrow$" When $\ell(M) \leq e$, since any two composition factors of $M$ are not isomorphic and $\operatorname{top}(M)$ is not a composition factor of the objects in $O_\nu(M)$ except $M$, it is clear that $O_\nu(M)$ is an orthogonal system in $A_n^\ell$-$\underline{\mathrm{mod}}$.

When $\ell+1-e \leq \ell(M) \leq \ell$, for any object $N$ in $O_\nu(M)$, consider the morphisms $f$ from $N$ to $\tau^{-e}(N)$ satisfying $\operatorname{Im} f = \operatorname{rad}^e(\tau^{-e}(N))$ and $g$ from $N$ to $N$ satisfying $\operatorname{Im} g = \operatorname{rad}^n(N)$. Notice that by Lemma 2.5, $\ell(N) = \ell(M)$. So by Lemma 2.6, $\underline{f} = 0$ and $\underline{g} = 0$. Furthermore, by Lemma 2.8, $\underline{\operatorname{Hom}}_A(N,N) \cong k$ and $\underline{\operatorname{Hom}}_A(N,\tau^{-e}(N)) = 0$ if $\tau^{-e}(N) \ncong N$. There is a similar proof for $N$ and $\tau^{-ke}(N)$ $(\tau^{-ke}(N) \ncong N, k \in \mathbb{N})$. Therefore $O_\nu(M)$ is an orthogonal system in $A_n^\ell$-$\underline{\mathrm{mod}}$ because of the arbitrariness of the module $N$.

"⇒" Consider the morphism $f$ from $M$ to $\tau^{-e}(M)$ satisfying $\operatorname{Im} f = \operatorname{rad}^e(\tau^{-e}(M))$. If $f = 0$, then $\ell(\tau^{-e}(M)) = \ell(M) \leq e$. If $f \neq 0$, then since $\underline{\operatorname{Hom}}_A(M, \tau^{-e}(M)) = 0$, by Lemma 2.6 we have $\ell + 1 - e \leq \ell(M) \leq \ell$. ■

For any symmetric Nakayama algebra, the Nakayama functor is isomorphic to the identity functor and therefore we have the following corollary.

COROLLARY 2.11. *Let $A_n^{dn} = kQ/I$ be a symmetric Nakayama algebra and $M = M_{j,t}^i$ an indecomposable non-projective $A_n^{dn}$-module. Then the following are equivalent:*

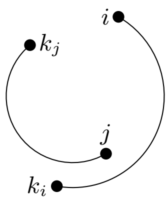
(1) $\underline{\operatorname{Hom}}_{A_n^{dn}}(M, M) \cong k$, *that is, $M$ is a stable brick.*
(2) $\ell(M) \leq n$ *or* $(d-1)n + 1 \leq \ell(M) \leq dn$.
(3) $t = 0$ *or* $t = d - 1$.

**3. Orthogonality for $A_n^{dn}$.** In this section, we introduce the arc for indecomposable $A_n^\ell$-modules and use it to describe orthogonality in the stable module category of any symmetric Nakayama algebra.

DEFINITION 3.1. Let $A_n^\ell = kQ/I$ be a self-injective Nakayama algebra. For any indecomposable $A_n^\ell$-module $M = M_{j,t}^i$, the *arc* of $M$ is defined to be the (unique) shortest path $\widehat{ij}$ from vertex $i$ to vertex $j$ in $Q$. In particular, if $i = j$, then the arc of $M$ is vertex $i$ in $Q$.

Notice that we also regard $Q$ as an oriented geometric graph, thus the arc of $M$ means the segment from $i$ to $j$ in $Q$. We now use the arc to describe the orthogonality relation between stable bricks over the symmetric Nakayama algebra $A_n^{dn}$.

LEMMA 3.2. *Let $M = M_{k_i,l_i}^i$ $(i \neq k_i, k_i - 1)$ and $N = N_{k_j,l_j}^j$ be stable bricks over $A_n^{dn}$. If their arcs intersect as follows (this means that $j \in \widehat{ik_i}$, $k_i \in \widehat{jk_j}$, $k_j \in \widehat{ji}$ and $k_j \neq i$):*




*then* $\underline{\operatorname{Hom}}_{A_n^{dn}}(N, M) \neq 0$.

*Proof.* If $l_i = 0$, then $\ell(M) \leq n$ and there exists a unique integer $t$ satisfying $\operatorname{top}(\operatorname{rad}^t(M)) \cong S_j$. Therefore, there is a morphism $f$ from $N$ to $M$ satisfying $\operatorname{Im} f = \operatorname{rad}^t(M)$ and the multiplicity of $S_i$ in $M/\operatorname{rad}^t(M)$ is 1. By Corollary 2.11, there are two cases for $l_j$:

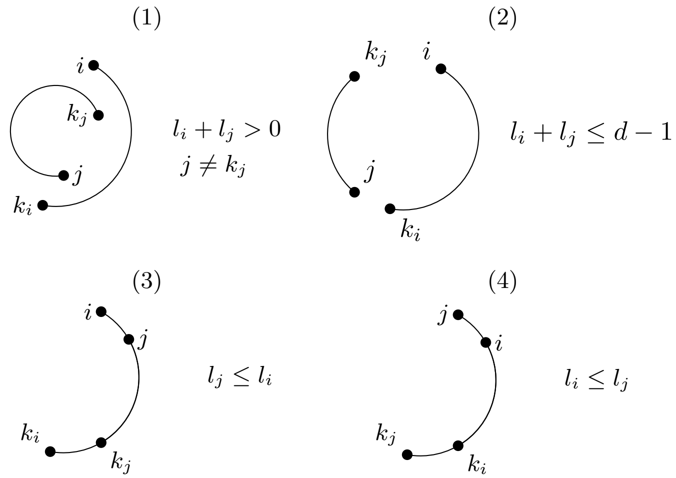
When $l_j = 0$, we can read off from the picture that $S_i$ is not a composition factor of $N$, and $\ell(N) + t \leq n < dn + 1$. By Lemma 2.6, $\underline{f} \neq 0$ and therefore $\underline{\mathrm{Hom}}_{A_n^{dn}}(N, M) \neq 0$. When $l_j = d-1$, the multiplicity of $S_i$ in $N$ is $d-1$, and $\ell(N) + t \leq dn < dn+1$. By Lemma 2.6, $\underline{f} \neq 0$ and therefore $\underline{\mathrm{Hom}}_{A_n^{dn}}(N, M) \neq 0$.

If $l_i = d-1$, then there is a minimum integer $t_1$ with $\mathrm{top}(\mathrm{rad}^{t_1}(M)) \cong S_j$ and a maximum integer $t_2$ satisfying $\mathrm{top}(\mathrm{rad}^{t_2}(M)) \cong S_j$. Again we consider two cases for $l_j$:

When $l_j = 0$, we also read off from the picture that $S_i$ is not a composition factor of $N$ and there is a morphism $f$ from $N$ to $M$ satisfying $\mathrm{Im}\, f = \mathrm{rad}^{t_2}(M)$, and $\ell(N) + t_2 \leq dn < dn+1$. By Lemma 2.6, $\underline{f} \neq 0$ and therefore $\underline{\mathrm{Hom}}_{A_n^{dn}}(N, M) \neq 0$. When $l_j = d-1$, there is a morphism $f$ from $N$ to $M$ satisfying $\mathrm{Im}\, f = \mathrm{rad}^{t_1}(M)$, and $\ell(N) + t_1 \leq dn < dn+1$. By Lemma 2.6, $\underline{f} \neq 0$ and therefore $\underline{\mathrm{Hom}}_{A_n^{dn}}(N, M) \neq 0$. ∎

PROPOSITION 3.3. *Let $A_n^{dn} = kQ/I$ $(d \geq 2)$ be a symmetric Nakayama algebra and let $M = M^i_{k_i,l_i}$, $N = N^j_{k_j,l_j}$ be indecomposable non-projective $A_n^{dn}$-modules. Then $\{M, N\}$ is an orthogonal system in $A_n^{dn}$-$\underline{\mathrm{mod}}$ if and only if it satisfies the following conditions:*

(a) *$i \neq j$ and $k_i \neq k_j$.*
(b) *$l_i = 0$ or $l_i = d-1$, while $l_j = 0$ or $l_j = d-1$.*
(c) *Their arcs belong to one of the four cases:*




*where the two arcs in (2) are disjoint and in the other cases the two arcs do intersect.*

*Proof.* "⇒" (a) and (b) follow from Lemma 2.9 and Corollary 2.11. The four pictures of arcs in (1)–(4) follow from Lemma 3.2; we just need to verify the conditions for $l_i$ and $l_j$ in four cases.

Case 1. If $l_i = l_j = 0$, then there is a unique integer $t$ satisfying $\mathrm{top}(\mathrm{rad}^t(M)) \cong S_j$ such that the multiplicity of $S_i$ in $M/\mathrm{rad}^t(M)$ is 1, and there is a morphism $f\colon N \to M$ satisfying $\mathrm{Im}\, f = \mathrm{rad}^t(M)$. Since the multiplicity of $S_i$ in $N$ is 1 and $d \geq 2$, by Remark 2.7 we have $\underline{f} \neq 0$. This contradiction shows that $l_i + l_j > 0$.

Case 2. If $l_i = d-1$, $l_j = d-1$, then the multiplicity of $S_i$ in $N$ is $d-1$ and the multiplicity of $S_j$ in $M$ is $d-1$. There exists a minimum integer $t$ satisfying $\mathrm{top}(\mathrm{rad}^t(M)) \cong S_j$ such that the multiplicity of $S_i$ in $M/\mathrm{rad}^t(M)$ is 1. There is a morphism $f\colon N \to M$ satisfying $\mathrm{Im}\, f = \mathrm{rad}^t(M)$. By Remark 2.7, $\underline{f} \neq 0$. This contradiction shows that $l_i + l_j \leq d-1$.

Case 3. If $l_i = 0$, $l_j = d-1$, then the multiplicity of $S_i$ in $N$ is $d-1$ and the multiplicity of $S_j$ in $M$ is 1. There exists a unique integer $t$ satisfying $\mathrm{top}(\mathrm{rad}^t(M)) \cong S_j$ such that the multiplicity of $S_i$ in $M/\mathrm{rad}^t(M)$ is 1, and there is a morphism $f\colon N \to M$ satisfying $\mathrm{Im}\, g = \mathrm{rad}^t(M)$. By Remark 2.7, $\underline{f} \neq 0$. This contradiction shows that $l_j \leq l_i$.

Case 4. If $l_i = d-1$, $l_j = 0$, then we can show similarly to Case 3 that $l_i \leq l_j$.

"$\Leftarrow$" By Corollary 2.11, we can assume that $M = M^i_{k_i,l_i}$ and $N = N^j_{k_j,l_j}$ are stable bricks in $A_n^{dn}$-$\underline{\mathrm{mod}}$ satisfying the following conditions: $i \neq j$, $k_i \neq k_j$, and $l_i$ is 0 or $d-1$, while $l_j$ is 0 or $d-1$.

We now prove that $\underline{\mathrm{Hom}}_{A_n^{dn}}(M, N) = 0$ and $\underline{\mathrm{Hom}}_{A_n^{dn}}(N, M) = 0$ by checking the four cases. In each case, we consider three subcases according to the values of $l_i$ and $l_j$.

Case 1. (i) When $l_i = 0$, $l_j = d-1$, the multiplicity of $S_i$ in $N$ is $d$ and the multiplicity of $S_j$ in $M$ is 1. There is a maximum integer $t_1$ satisfying $\mathrm{top}(\mathrm{rad}^{t_1}(N)) \cong S_i$ such that the multiplicity of $S_j$ in $N/\mathrm{rad}^{t_1}(N)$ is $d$. There exists a morphism $f\colon M \to N$ satisfying $\mathrm{Im}\, f = \mathrm{rad}^{t_1}(N)$. By Remark 2.7, $\underline{f} = 0$, and by Lemma 2.8, $\underline{\mathrm{Hom}}_{A_n^{dn}}(M, N) = 0$. Moreover, there is a unique integer $t_2$ satisfying $\mathrm{top}(\mathrm{rad}^{t_2}(M)) \cong S_j$ and a morphism $g\colon N \to M$ satisfying $\mathrm{Im}\, g = \mathrm{rad}^{t_2}(M)$. By Remark 2.7, $\underline{g} = 0$, and by Lemma 2.8, $\underline{\mathrm{Hom}}_{A_n^{dn}}(N, M) = 0$.

(ii) When $l_i = d-1$, $l_j = 0$, the multiplicity of $S_i$ in $N$ is 1 and the multiplicity of $S_j$ in $M$ is $d$. There is a description similar to (i) for this case, and we have $\underline{\mathrm{Hom}}_{A_n^{dn}}(N, M) = 0$ and $\underline{\mathrm{Hom}}_{A_n^{dn}}(M, N) = 0$.

(iii) When $l_i = d-1$, $l_j = d-1$, the multiplicity of $S_i$ in $N$ is $d$ and the multiplicity of $S_j$ in $M$ is $d$. There is a minimum integer $t_1$ satisfying $\mathrm{top}(\mathrm{rad}^{t_1}(N)) \cong S_i$ such that the multiplicity of $S_j$ in $N/\mathrm{rad}^{t_1}(N)$ is 1. There exists a morphism $f\colon M \to N$ satisfying $\mathrm{Im}\, f = \mathrm{rad}^{t_1}(N)$. By Remark 2.7, $\underline{f} = 0$, and by Lemma 2.8, $\underline{\mathrm{Hom}}_{A_n^{dn}}(M, N) = 0$. Similarly, there is a minimum integer $t_2$ satisfying $\mathrm{top}(\mathrm{rad}^{t_2}(M)) \cong S_j$ such that the multiplicity of $S_i$

in $M/\mathrm{rad}^{t_2}(M)$ is 1. There is a morphism $g\colon N \to M$ satisfying $\mathrm{Im}\, g = \mathrm{rad}^{t_2}(M)$. By Remark 2.7, $\underline{g} = 0$, and by Lemma 2.8, $\underline{\mathrm{Hom}}_{A_n^{dn}}(N, M) = 0$.

CASE 2. (i) When $l_i=0$, $l_j=0$, $S_j$ is not a composition factor of $M$ and $S_i$ is not a composition factor of $N$. Then $\mathrm{Hom}_{A_n^{dn}}(M,N)=0$, $\mathrm{Hom}_{A_n^{dn}}(N,M) =0$ and therefore $\underline{\mathrm{Hom}}_{A_n^{dn}}(M,N) = 0$, $\underline{\mathrm{Hom}}_{A_n^{dn}}(N,M) = 0$.

(ii) When $l_i = 0$, $l_j = d-1$, $S_j$ is not a composition factor of $M$ and the multiplicity of $S_i$ in $N$ is $d-1$. Then $\mathrm{Hom}_{A_n^{dn}}(N,M) = 0$ and there is a maximum integer $t$ satisfying $\mathrm{top}(\mathrm{rad}^t(N)) \cong S_i$, but $\ell(\mathrm{rad}^t(N)) > \ell(M)$, and we have $\mathrm{Hom}_{A_n^{dn}}(M,N) = 0$. Therefore, $\underline{\mathrm{Hom}}_{A_n^{dn}}(M,N) = 0$ and $\underline{\mathrm{Hom}}_{A_n^{dn}}(N,M) = 0$.

(iii) When $l_i = d-1$, $l_j = 0$, $S_i$ is not a composition factor of $N$ and the multiplicity of $S_j$ in $M$ is $d-1$. It follows much as above that $\underline{\mathrm{Hom}}_{A_n^{dn}}(M,N) = 0$, $\underline{\mathrm{Hom}}_{A_n^{dn}}(N,M) = 0$.

CASE 3. (i) When $l_i = 0$, $l_j = 0$, $S_i$ is not a composition factor of $N$. Then $\mathrm{Hom}_{A_n^{dn}}(M,N) = 0$ and therefore $\underline{\mathrm{Hom}}_{A_n^{dn}}(M,N) = 0$. There is a unique integer $t$ satisfying $\mathrm{top}(\mathrm{rad}^t(M)) \cong S_j$, but $\ell(\mathrm{rad}^t(M)) > \ell(N)$, so we have $\mathrm{Hom}_{A_n^{dn}}(N,M) = 0$ and therefore $\underline{\mathrm{Hom}}_{A_n^{dn}}(N,M) = 0$.

(ii) When $l_i = d-1$, $l_j = 0$, $S_i$ is not a composition factor of $N$. Then $\mathrm{Hom}_{A_n^{dn}}(M,N) = 0$ and therefore $\underline{\mathrm{Hom}}_{A_n^{dn}}(M,N) = 0$. There is a maximum integer $t$ satisfying $\mathrm{top}(\mathrm{rad}^t(M)) \cong S_j$, but $\ell(\mathrm{rad}^t(M)) > \ell(N)$. Then $\mathrm{Hom}_{A_n^{dn}}(N,M) = 0$ and $\underline{\mathrm{Hom}}_{A_n^{dn}}(N,M) = 0$.

(iii) When $l_i = d-1$, $l_j = d-1$, the multiplicity of $S_i$ in $N$ is $d-1$ and the multiplicity of $S_j$ in $M$ is $d$. There exists a minimum integer $t_1$ satisfying $\mathrm{top}(\mathrm{rad}^{t_1}(N)) \cong S_i$ such that the multiplicity of $S_j$ in $N/\mathrm{rad}^{t_1}(N)$ is 1, and there is a morphism $f\colon M \to N$ satisfying $\mathrm{Im}\, f = \mathrm{rad}^{t_1}(N)$. By Remark 2.7, $\underline{f} = 0$, and by Lemma 2.8, $\underline{\mathrm{Hom}}_{A_n^{dn}}(M,N) = 0$. There exists an integer $t_2$ satisfying $\mathrm{top}(\mathrm{rad}^{t_2}(M)) \cong S_j$ such that the multiplicity of $S_i$ in $M/\mathrm{rad}^{t_2}(M)$ is 2, and there is a morphism $g\colon N \to M$ satisfying $\mathrm{Im}\, g = \mathrm{rad}^{t_2}(M)$. By Remark 2.7, $\underline{g} = 0$, and by Lemma 2.8, $\underline{\mathrm{Hom}}_{A_n^{dn}}(N,M) = 0$.

CASE 4. This is similar to Case 3.

Summarizing the above discussion we see that $\{M, N\}$ is an orthogonal system in $A_n^{dn}$-$\underline{\mathrm{mod}}$. ∎

REMARK 3.4. When $d = 1$, the assertion of Proposition 3.3 remains valid without the conditions for $l_i$ and $l_j$ in (c).

## 4. A construction of sms's over $A_n^{\ell}$

**4.1. Non-crossing partitions.** In this subsection, we first introduce (classical) non-crossing partitions, and then we give some observations on the non-crossing partitions associated to sms's over $A_n^{dn}$.

Definition 4.1 (cf. [10]). A *partition* of the set $\underline{n} := \{1, \ldots, n\}$ is a map $p$ from $\underline{n}$ to its power set with the following properties: (1) $i \in p(i)$ for all $1 \le i \le n$; (2) $p(i) = p(j)$ or $p(i) \cap p(j) = \emptyset$ for all $1 \le i, j \le n$. We call $p(i)$ a *block* of $p$. A *non-crossing partition* of $\underline{n}$ is a partition $p$ such that no two blocks cross each other, that is, if $a$ and $b$ belong to one block and $x$ and $y$ belong to another, we cannot have $a < x < b < y$.

We show how an sms $\mathcal{S}$ of $A_n^{dn}$ relates to a non-crossing partition. By [9, Proposition 6.2], both the top and the socle series of $\mathcal{S}$ give a complete set $\{S_1, \ldots, S_n\}$ of simple $A_n^{dn}$-modules. For each $1 \le i \le n$, there is a subset $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ of $\underline{n}$ such that there exists an object $M_{ij}$ in $\mathcal{S}$ with $\mathrm{top}(M_{ij}) \cong S_{k_i^{(j)}}$, $\mathrm{soc}(M_{ij}) \cong S_{k_i^{(j+1)}}$ for each $0 \le j \le s_i - 1$, where $k_i^{(0)} = k_i^{(s_i)} = i$, $k_i^{(1)} = k_i$. In this way, we get a partition $p$ of $\underline{n}$.

Remark 4.2. Since we have the subset $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ of $\underline{n}$ for each $1 \le i \le n$, we can define a permutation $\sigma$ of $\underline{n}$ such that $\sigma(i) = k_i$ for any $i$ in $\underline{n}$. Moreover, $\sigma^j(i) = k_i^{(j)}$ for each $2 \le j \le s_i - 1$.

Example 4.3. Consider

$$\mathcal{S} = \left\{ 4\,,\ \begin{matrix} 2 \\ 3 \\ 4 \\ 1 \end{matrix}\,,\ \begin{matrix} 3 \\ 4 \\ 1 \\ 2 \end{matrix}\,,\ \begin{matrix} 1 \\ 2 \\ 3 \end{matrix} \right\}$$

in $A_4^4$-$\underline{\mathrm{mod}}$. Here we use Loewy diagrams to specify indecomposable modules for simplicity. By Theorem 2.3, $\mathcal{S}$ is an sms of $A_4^4$. By the definition of $p(i)$, we have $p(1) = p(2) = p(3) = \{3, 2, 1\}$ and $p(4) = \{4\}$. Moreover, the permutation $\sigma$ of $\underline{4}$ defined in Remark 4.2 is as follows: $\sigma(1) = 3$, $\sigma(2) = 1$, $\sigma(3) = 2$ and $\sigma(4) = 4$.

From now on we fix the following notations: $\mathcal{S}$ is an sms of $A_n^{dn}$, and $p$ is the corresponding partition. For each $1 \le i \le n$, the block $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ is as explained above, that is, there exists an object $M_{ij}$ in $\mathcal{S}$ satisfying $\mathrm{top}(M_{ij}) \cong S_{k_i^{(j)}}$ and $\mathrm{soc}(M_{ij}) \cong S_{k_i^{(j+1)}}$ for each $0 \le j \le s_i - 1$, where $k_i^{(0)} = k_i^{(s_i)} = i$, $k_i^{(1)} = k_i$.

By Proposition 3.3, the partition $p$ has the following "anti-clockwise" property.

Corollary 4.4. *Let $\mathcal{S}$ be an sms of $A_n^{dn} = kQ/I$ and $p$ the partition obtained as above. Suppose that $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$, where $s_i \ge 3$. Then $k_i^{(t)}$ is a vertex on the arc $\widehat{ik_i^{(t-1)}}$ in the quiver $Q$ for each $2 \le t \le s_i - 1$.*

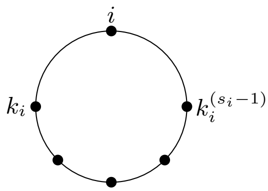
*Proof.* First, we consider the objects $M_{i0}$ and $M_{i1}$ in $\mathcal{S}$, where $\{M_{i0}, M_{i1}\}$ is an orthogonal system in $A_n^{dn}$-$\underline{\mathrm{mod}}$ and $\mathrm{top}(M_{i0}) \cong S_i$, $\mathrm{soc}(M_{i0}) \cong S_{k_i}$,

$\mathrm{top}(M_{i1}) \cong S_{k_i}$, $\mathrm{soc}(M_{i1}) \cong S_{k_i^{(2)}}$. Their arcs must be as in case (1) of Proposition 3.3(c). Then $k_i^{(2)}$ is a vertex on the arc $\widehat{ik_i}$ from vertex $i$ to vertex $k_i$. Similarly, when $s_i \geq 4$, $k_i^{(t)}$ is a vertex on the arc $\widehat{ik_i^{(t-1)}}$ for each $3 \leq t \leq s_i - 1$. ∎

We are ready to prove that the above partition $p$ corresponding to $\mathcal{S}$ is actually a non-crossing partition.

Corollary 4.5. *Let $\mathcal{S}$ be an sms of $A_n^{dn}$ and $p$ the partition corresponding to $\mathcal{S}$. Then $p$ is a non-crossing partition of $\underline{n}$.*

*Proof.* With the above notations, $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ and there exists an object $M_{ij}$ in $\mathcal{S}$ satisfying $\mathrm{top}(M_{ij}) \cong S_{k_i^{(j)}}$ and $\mathrm{soc}(M_{ij}) \cong S_{k_i^{(j+1)}}$ for each $0 \leq j \leq s_i - 1$, where $k_i^{(0)} = k_i^{(s_i)} = i$, $k_i^{(1)} = k_i$. Take two different blocks $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ and $p(j) = \{j, k_j, k_j^{(2)}, \ldots, k_j^{(s_j-1)}\}$. By Corollary 4.4, we have the following graph of vertices in $p(i)$:




Without loss of generality we can assume that vertex $j$ lies on the arc $\widehat{k_i i}$. We claim that $k_j$, $k_j^{(2)}, \ldots,$ $k_j^{(s_j-1)}$ are also vertices on $\widehat{k_i i}$. Otherwise, without loss of generality we can assume that $k_j$ is a vertex on $\widehat{ik_i}$. Moreover, there exist objects $M_{i0}$ in $\mathcal{S}$ satisfying $\mathrm{top}(M_{i0}) \cong S_i$, $\mathrm{soc}(M_{i0}) \cong S_{k_i}$ and $M_{j0}$ in $\mathcal{S}$ satisfying $\mathrm{top}(M_{j0}) \cong S_j$, $\mathrm{soc}(M_{j0}) \cong S_{k_j}$. By Lemma 3.2, $\underline{\mathrm{Hom}}_{A_n^{dn}}(M_{i0}, M_{j0}) \neq 0$. This is a contradiction.

Therefore, $p$ is a non-crossing partition. ∎

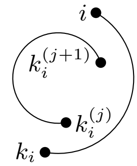
In the next two results, we use non-crossing partitions to describe some properties of sms's.

Lemma 4.6. *Let $\mathcal{S}$ be an sms of $A_n^{dn}$ $(d \geq 2)$ and $p$ the corresponding non-crossing partition. For each $1 \leq i \leq n$, let $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ be as above. For each $0 \leq j \leq s_i - 1$, assume that $M_{ij} = M^{k_i^{(j)}}_{k_i^{(j+1)}, l_{ij}}$ for some $l_{ij} \geq 0$. Then there is at most one $l_{ij}$ satisfying $l_{ij} = 0$ for all $0 \leq j \leq s_i - 1$.*

*Proof.* If $s_i = 1$, then $p(i) = \{i\}$ and the desired result follows.

If $s_i \geq 2$, without loss of generality we can assume $l_{i0} = 0$. When $s_i \geq 3$, we use Corollary 4.4. Notice that $k_i^{(2)} = i$ when $s_i = 2$. Then, regardless of whether $s_i \geq 3$ or $s_i = 2$, the arcs of $M_{i0}$ and $M_{ij}$ are as follows for any

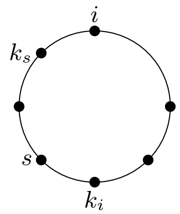
$1 \leq j \leq s_i - 1$:




Since $\{M_{i0}, M_{ij}\}$ is an orthogonal system in $A_n^{dn}$-$\underline{\mathrm{mod}}$, by Proposition 3.3 we must have $l_{ij} = d-1$ for each $1 \leq j \leq s_i - 1$. ∎

LEMMA 4.7. *Let $\mathcal{S}$ be an sms of $A_n^{dn}$ $(d \geq 2)$ and $p$ the corresponding non-crossing partition. For each $1 \leq i \leq n$, let $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ be as above. For each $0 \leq j \leq s_i - 1$, assume that $M_{ij} = M^{k_i^{(j)}}_{k_i^{(j+1)}, l_{ij}}$ for some $l_{ij} \geq 0$. Suppose that there exists some $i$ satisfying $l_{ij} = d-1$ for all $0 \leq j \leq s_i - 1$. Then, for any block $p(s)$ different from $p(i)$ there is only one $t$ such that $l_{st} = 0$.*

*Proof.* Without loss of generality we can assume the vertices in $p(i)$ and in $p(s)$ are located as follows:




Consider the modules $M^s_{k_s, l_{s0}}$ and $M^i_{k_i, l_{i0}}$. Since $\{M^s_{k_s, l_{s0}}, M^i_{k_i, l_{i0}}\}$ is an orthogonal system in $A_n^{dn}$-$\underline{\mathrm{mod}}$, by Proposition 3.3 we must have $l_{s0} = 0$. Moreover, by Lemma 4.6, $l_{s0}$ is unique. ∎

**4.2. The construction of sms's.** In this subsection, we give an explicit construction of sms's over any self-injective Nakayama algebra. We first construct sms's of a symmetric Nakayama algebra and then use covering theory to deal with the general case.

We denote by $\mathcal{P}$ the set of non-crossing partitions of $\underline{n} = \{1, \ldots, n\}$ and given $i \in \mathbb{Z}$, let $\overline{i}$ be the positive integer in $\underline{n}$ such that $i \equiv \overline{i} \bmod n$. For $p \in \mathcal{P}$, let $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ (with the ordering as in Corollary 4.4, when $s_i \geq 3$) be the block for $1 \leq i \leq n$ and let $\widehat{i}$ be the set $\{i, \overline{i+1}, \ldots, k_i\}$. With these notations, we introduce the following definition.

DEFINITION 4.8. Let $A_n^{dn}$ be a symmetric Nakayama algebra and $\mathcal{P}$ the set of non-crossing partitions of $\underline{n}$, where $n, d$ are positive integers. For any $p \in \mathcal{P}$ and any $1 \leq k \leq n$, we define two types of sets of indecomposable

$A_n^{dn}$-modules:

$$\mathcal{L}_{p,k} = \{M^i_{k_i,l_i} \mid i = 1, \ldots, n\}, \quad \text{where} \quad l_i = \begin{cases} 0, & \widehat{i} \cap p(k) = \emptyset, \\ d-1, & \textit{otherwise}; \end{cases}$$

$$\mathcal{S}_{p,k} = \{M^i_{k_i,l_i} \mid i = 1, \ldots, n\}, \quad \text{where} \quad l_i = \begin{cases} 0, & \widehat{i} \cap p(k) = \emptyset, \\ 0, & i = k, \\ d-1, & \textit{otherwise}. \end{cases}$$

REMARK 4.9. From the above definition, the cardinalities of $\mathcal{L}_{p,k}$ and $\mathcal{S}_{p,k}$ are equal to the number of non-isomorphic simple $A_n^{dn}$-modules. If $d \geq 2$, we have the following facts about $\mathcal{L}_{p,k}$ and $\mathcal{S}_{p,k}$. Let $p \in \mathcal{P}$ and $1 \leq k \leq n$. The modules $M^i_{k_i,l_i}$ with $i \in p(k)$ in $\mathcal{L}_{p,k}$ satisfy $l_i = d-1$, and for each block $p(t)$ different from $p(k)$ there exists a unique module $M^i_{k_i,l_i}$ in $\mathcal{L}_{p,k}$ satisfying $i \in p(t)$ and $l_i = 0$. Moreover, for each block $p(t)$, there exists a unique module $M^i_{k_i,l_i}$ in $\mathcal{S}_{p,k}$ satisfying $i \in p(t)$ and $l_i = 0$.

THEOREM 4.10. *Let $A_n^{dn}$ be a symmetric Nakayama algebra and $\mathcal{P}$ the set of non-crossing partitions of $\underline{n}$. Then we have the following:*

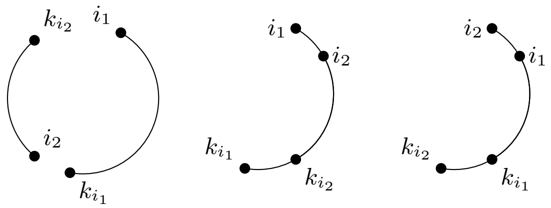
(a) *For any $p \in \mathcal{P}$ and any $1 \leq k \leq n$, $\mathcal{L}_{p,k}$ and $\mathcal{S}_{p,k}$ are sms's.*
(b) *All sms's of $A_n^{dn}$ are of these forms.*
(c) *If $d \geq 2$, then for $p, p' \in \mathcal{P}$ and $1 \leq k, k' \leq n$, we have the following results:*
    (1) $\mathcal{L}_{p,k} \neq \mathcal{S}_{p',k'}$.
    (2) $\mathcal{L}_{p,k} = \mathcal{L}_{p',k'}$ *if and only if $p = p'$ and $p(k) = p(k')$.*
    (3) $\mathcal{S}_{p,k} = \mathcal{S}_{p',k'}$ *if and only if the following three conditions hold:* (i) $p = p'$; (ii) $k = k'$ *or* $\widehat{k} \cap \widehat{k'} = \emptyset$; (iii) $p_{k \vee k'} \in \mathcal{P}$, *where*

$$p_{k \vee k'}(i) = \begin{cases} p(k) \cup p(k'), & i \in p(k) \cup p(k'), \\ p(i), & \textit{otherwise}. \end{cases}$$

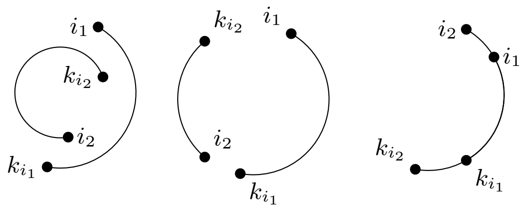
*Proof.* (a) We only prove that $\mathcal{L}_{p,k}$ is an sms, since the proof for $\mathcal{S}_{p,k}$ is similar. Since the Nakayama functor $\nu$ is isomorphic to the identity functor, $\mathcal{L}_{p,k}$ is Nakayama-stable. By Theorem 2.3, it is enough to show that any two objects $M^{i_1}_{k_{i_1},l_{i_1}}$ and $M^{i_2}_{k_{i_2},l_{i_2}}$ ($i_1 \neq i_2$) in $\mathcal{L}_{p,k}$ ($p \in \mathcal{P}, 1 \leq k \leq n$) form an orthogonal system in $A_n^{dn}$-$\underline{\text{mod}}$. When $d = 1$, $l_{i_1} = l_{i_2} = 0$, since $p$ is a non-crossing partition, there are four cases for the arcs of $M^{i_1}_{k_{i_1},l_{i_1}}$ and $M^{i_2}_{k_{i_2},l_{i_2}}$ corresponding to the four diagrams of Proposition 3.3. It follows from Remark 3.4 that $\{M^{i_1}_{k_{i_1},l_{i_1}}, M^{i_2}_{k_{i_2},l_{i_2}}\}$ is an orthogonal system in $A_n^{dn}$-$\underline{\text{mod}}$.

When $d \geq 2$, by the definition of $\mathcal{L}_{p,k}$, we consider four cases (1)–(4). In each case it is straightforward to check by Proposition 3.3 that $\{M^{i_1}_{k_{i_1},l_{i_1}}, M^{i_2}_{k_{i_2},l_{i_2}}\}$ is an orthogonal system in $A_n^{dn}$-$\underline{\text{mod}}$. We now list all the cases:

(1) $l_{i_1} = l_{i_2} = 0$, that is, $\widehat{i_1} \cap p(k) = \emptyset$, $\widehat{i_2} \cap p(k) = \emptyset$. There are three subcases for the arcs of $M^{i_1}_{k_{i_1}, l_{i_1}}$ and $M^{i_2}_{k_{i_2}, l_{i_2}}$:

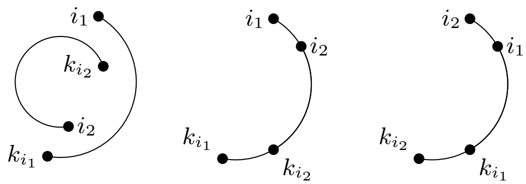



(2) $l_{i_1} = 0$, $l_{i_2} = d-1$, that is, $\widehat{i_1} \cap p(k) = \emptyset$, $\widehat{i_2} \cap p(k) \neq \emptyset$. There are three subcases for the arcs of $M^{i_1}_{k_{i_1}, l_{i_1}}$ and $M^{i_2}_{k_{i_2}, l_{i_2}}$:




(3) $l_{i_1} = d-1$, $l_{i_2} = 0$, that is, $\widehat{i_1} \cap p(k) \neq \emptyset$, $\widehat{i_2} \cap p(k) = \emptyset$. This is similar to Case (2).

(4) $l_{i_1} = l_{i_2} = d-1$, that is, $\widehat{i_1} \cap p(k) \neq \emptyset$, $\widehat{i_2} \cap p(k) \neq \emptyset$. There are three subcases for the arcs of $M^{i_1}_{k_{i_1}, l_{i_1}}$ and $M^{i_2}_{k_{i_2}, l_{i_2}}$:




(b) By Corollary 4.5, any sms $\mathcal{S}$ of $A_n^{dn}$ determines a non-crossing partition $p$ in $\mathcal{P}$. For $1 \leq i \leq n$, we denote by $p(i)$ the block which $i$ belongs to. Then we can assume that $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i - 1)}\}$ and there exists $M_{ij}$ in $\mathcal{S}$ satisfying $\mathrm{top}(M_{ij}) \cong S_{k_i^{(j)}}$ and $\mathrm{soc}(M_{ij}) \cong S_{k_i^{(j+1)}}$ for each $0 \leq j \leq s_i - 1$, where $k_i^{(0)} = k_i^{(s_i)} = i, k_i^{(1)} = k_i$. Notice that with our notation, $M_{ij} = M^{k_i^{(j)}}_{k_i^{(j+1)}, l_{ij}}$ for $0 \leq j \leq s_i - 1$ (cf. Lemmas 4.6 and 4.7).

If there is a block $p(i)$ satisfying $l_{ij} = d-1$ for each $0 \leq j \leq s_i - 1$, then from the proof of Lemma 4.7, we know for each block $p(s)$ that

$$l_{st} = \begin{cases} 0, & \widehat{k_s^{(t)}} \cap p(i) = \emptyset, \\ d-1, & \text{otherwise.} \end{cases}$$

Therefore $\mathcal{S} = \mathcal{L}_{p,i}$.

If there is no block $p(i)$ satisfying $l_{ij} = d-1$ for each $0 \leq j \leq s_i - 1$, suppose that $p(i_1), \ldots, p(i_k)$ are all blocks; by Lemma 4.6, without loss of generality, we assume $l_{i_t 0} = 0$ for any block $p(i_t)$. We have $l_{i_t j} = d-1$ for $j \neq 0$. Then there exists some $i$ in $\{i_1, \ldots, i_k\}$ satisfying $\widehat{i_t} \cap p(i) = \emptyset$ for any $i_t \neq i$. It follows easily that $\mathcal{S}$ must have the form $\mathcal{S}_{p,i}$.

(c) (1) This follows easily from Remark 4.9. More precisely, if $d \geq 2$, then for each block $p'(t)$ there exists a unique $M^i_{k_i,l_i}$ in $\mathcal{S}_{p',k'}$ with $i \in p'(t)$ and $l_i = 0$; however, all the modules $M^i_{k_i,l_i}$ in $\mathcal{L}_{p,k}$ which correspond to the block $p(k)$ satisfy $l_i = d-1$ for any $i$ in this block. Therefore $\mathcal{L}_{p,k} \neq \mathcal{S}_{p',k'}$ for any $p, p', k, k'$.

(2) If $p = p'$, $p(k) = p(k')$, then by the definitions of $\mathcal{L}_{p,k}$ and $\mathcal{L}_{p,k'}$ we have $\mathcal{L}_{p,k} = \mathcal{L}_{p',k'}$.

If $\mathcal{L}_{p,k} = \mathcal{L}_{p',k'}$, then $p = p'$. Otherwise, there exist $M^i_{k_i,l_i}$ in $\mathcal{L}_{p,k}$ and $M^i_{k'_i,l'_i}$ in $\mathcal{L}_{p',k'}$ with $k_i \neq k'_i$, and therefore $M^i_{k_i,l_i} \neq M^i_{k'_i,l'_i}$, which contradicts $\mathcal{L}_{p,k} = \mathcal{L}_{p',k'}$. Assume now that $\mathcal{L}_{p,k} = \mathcal{L}_{p,k'}$. We have $l_i = d-1$ for $M^i_{k_i,l_i}$ in $\mathcal{L}_{p,k}$, where $i \in p(k)$ or $i \in p(k')$. By Lemma 4.7, there is only one such block for $\mathcal{L}_{p,k}$. Therefore $p(k) = p(k')$.

(3) If $\mathcal{S}_{p,k} = \mathcal{S}_{p',k'}$, then $p = p'$. Otherwise, there exist $M^i_{k_i,l_i} \in \mathcal{S}_{p,k}$ and $M^i_{k'_i,l'_i} \in \mathcal{S}_{p',k'}$ with $k_i \neq k'_i$, and therefore $M^i_{k_i,l_i} \neq M^i_{k'_i,l'_i}$, contradicting $\mathcal{S}_{p,k} = \mathcal{S}_{p',k'}$. Assume now that $\mathcal{S}_{p,k} = \mathcal{S}_{p,k'}$ and $k \neq k'$. By the definition of $\mathcal{S}_{p,k}$, $\mathcal{S}_{p,k} = \mathcal{S}_{p,k'}$ if and only if the following conditions hold:

- $\widehat{k} \cap p(k') = \emptyset$;
- $\widehat{k'} \cap p(k) = \emptyset$;
- $\widehat{i} \cap p(k) = \emptyset$ if and only if $\widehat{i} \cap p(k') = \emptyset$ for $i \neq k, k'$.

The first two conditions are equivalent to $\widehat{k} \cap \widehat{k'} = \emptyset$. Moreover, the last condition implies that $p_{k \vee k'}$ is also a non-crossing partition. Conversely, if $p_{k \vee k'}$ is a non-crossing partition, then clearly the last condition holds. ∎

Remark 4.11. (1) Notice that the partition associated with the sms $\mathcal{S}_{p,k}$ or $\mathcal{L}_{p,k}$ (as discussed in Subsection 4.1) is exactly $p$.

(2) For an equivalent formulation of the condition $\mathcal{S}_{p,k} = \mathcal{S}_{p',k'}$, see Remark 5.6.

(3) For $1 \leq k, k' \leq n$, if $d = 1$, then $\mathcal{L}_{p,k} = \mathcal{L}_{p,k'} = \mathcal{S}_{p,k} = \mathcal{S}_{p,k'}$.

Example 4.12. We describe the sms's of the symmetric Nakayama algebra $A^6_2$ using the set $\mathcal{P}$ of non-crossing partitions of $\underline{2}$. Since $\mathcal{P} = \{p_1, p_2\}$ where $p_1 = \{\{1\}, \{2\}\}$, $p_2 = \{\{1, 2\}\}$, we can directly write down all sms's of $A_2^6$ from the definitions of $\mathcal{L}_{p,k}$ and $\mathcal{S}_{p,k}$:

$$\mathcal{L}_{p_1,1} = \{M_{1,2}^1, M_{2,0}^2\} = \left\{ \begin{matrix} 1\\2\\1\\2\\1 \end{matrix}\,,\ 2 \right\}, \quad \mathcal{L}_{p_1,2} = \{M_{1,0}^1, M_{2,2}^2\} = \left\{ 1\,,\ \begin{matrix} 2\\1\\2\\1\\2 \end{matrix} \right\},$$

$$\mathcal{S}_{p_2,1} = \{M_{2,0}^1, M_{1,2}^2\} = \left\{ \begin{matrix} 1\\2 \end{matrix}\,,\ \begin{matrix} 2\\1\\2\\1\\2\\1 \end{matrix} \right\}, \quad \mathcal{S}_{p_2,2} = \{M_{2,2}^1, M_{1,0}^2\} = \left\{ \begin{matrix} 1\\2\\1\\2\\1\\2 \end{matrix}\,,\ \begin{matrix} 2\\1 \end{matrix} \right\},$$

$$\mathcal{L}_{p_2,1} = \mathcal{L}_{p_2,2} = \{M_{2,2}^1, M_{1,2}^2\} = \left\{ \begin{matrix} 1\\2\\1\\2\\1\\2 \end{matrix}\,,\ \begin{matrix} 2\\1\\2\\1\\2\\1 \end{matrix} \right\},$$

$$\mathcal{S}_{p_1,1} = \mathcal{S}_{p_1,2} = \{M_{1,0}^1, M_{2,0}^2\} = \{1, 2\}.$$

In the following, using covering theory, we describe the sms's of a self-injective Nakayama algebra $A_n^\ell$. We first recall some notions.

Definition 4.13 ([3, Definition 1.3]). A translation-quiver morphism $f\colon \Delta \to \Gamma$ is called a *covering* if for each point $p \in \Delta_0$ the induced maps $p^- \to f(p)^-$ and $p^+ \to f(p)^+$ are bijections. Furthermore, $\tau(p)$ and $\tau^-(q)$ should be defined if so are $\tau(f(p))$ and $\tau^-(f(q))$ respectively (of course, since $f$ is a translation-quiver morphism, we have $f(\tau(p)) = \tau(f(p))$ whenever $\tau(p)$ is defined).

Definition 4.14 ([3, Definition 3.1]). Let $F : \mathcal{C} \to \mathcal{D}$ be a $k$-linear functor between two $k$-categories. Then $F$ is called a *covering functor* if the maps

$$\coprod_{z/b} \mathcal{C}(x, z) \to \mathcal{D}(a, b) \quad \text{and} \quad \coprod_{t/a} \mathcal{C}(t, y) \to \mathcal{D}(a, b),$$

induced by $F$, are bijective for any two objects $a$ and $b$ of $\mathcal{D}$. Here $t$ and $z$ range over all objects of $\mathcal{C}$ such that $Ft = a$ and $Fz = b$ respectively; the maps are supposed to be bijective for all $x$ and $y$ chosen among the $t$ and $z$ respectively.

Lemma 4.15. *Let $A = A_n^\ell$ and $B = A_e^\ell$ be two self-injective Nakayama algebras such that $e$ is the greatest common divisor of $n$ and $\ell$. Then there is* *a covering of stable translation quivers* $\pi\colon {}_s\Gamma_A \to {}_s\Gamma_B \cong {}_s\Gamma_A/\langle\nu\rangle$ *(where* $\nu$ *is the Nakayama automorphism of* ${}_s\Gamma_A$*), which induces a covering functor* $F\colon A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$*. Consequently, if* $\mathcal{S}$ *is an orthogonal system in* $B$-$\underline{\mathrm{mod}}$*, then* $\mathcal{S}$ *is an sms of* $B$-$\underline{\mathrm{mod}}$ *if and only if* $F^{-1}(\mathcal{S})$ *is an sms of* $A$-$\underline{\mathrm{mod}}$*. Moreover, if* $\mathcal{S}'$ *is a Nakayama-stable orthogonal system in* $A$-$\underline{\mathrm{mod}}$*, then* $F^{-1}(F(\mathcal{S}')) = \mathcal{S}'$ *and therefore* $\mathcal{S}'$ *is an sms of* $A$-$\underline{\mathrm{mod}}$ *if and only if* $F(\mathcal{S}')$ *is an sms of* $B$-$\underline{\mathrm{mod}}$*. In particular, there is a bijection between the sms's of* $A$-$\underline{\mathrm{mod}}$ *and the sms's of* $B$-$\underline{\mathrm{mod}}$ *induced by* $F$*.*

*Proof.* Clearly, there is a covering of stable translation quivers $\pi\colon {}_s\Gamma_A \to {}_s\Gamma_B \cong {}_s\Gamma_A/\langle\nu\rangle$, where $\nu$ is the Nakayama automorphism of ${}_s\Gamma_A$. It follows that there is a covering functor between the corresponding mesh categories (see [12, Section 2]) $k({}_s\Gamma_A)$ and $k({}_s\Gamma_B)$. On the other hand, since $A$ and $B$ are standard representation-finite self-injective algebras (see [1, Section 2]), we can identify $k({}_s\Gamma_A)$ and $k({}_s\Gamma_B)$ with $A$-$\underline{\mathrm{ind}}$ and $B$-$\underline{\mathrm{ind}}$, respectively (see [6, Section 3]). Therefore we get a covering functor $A\text{-}\underline{\mathrm{ind}} \to B\text{-}\underline{\mathrm{ind}}$, which extends to a covering functor $F : A\text{-}\underline{\mathrm{mod}} \to B\text{-}\underline{\mathrm{mod}}$ such that $F^{-1}(Y)$ is the $\nu$-orbit of $X$ for any object $Y$ in $B$-$\underline{\mathrm{ind}}$, where $F(X) = Y$ for some object $X$ in $A$-$\underline{\mathrm{ind}}$. Hence, for an orthogonal system $\mathcal{S}$ in $B$-$\underline{\mathrm{mod}}$, $\mathcal{S}$ is an sms of $B$-$\underline{\mathrm{mod}}$ if and only if $F^{-1}(\mathcal{S})$ is an sms of $A$-$\underline{\mathrm{mod}}$. Notice that $F(F^{-1}(\mathcal{S})) = \mathcal{S}$.

We next show $F^{-1}(F(\mathcal{S}')) = \mathcal{S}'$ for any Nakayama-stable orthogonal system $\mathcal{S}'$ in $A$-$\underline{\mathrm{mod}}$. It is easy to see $\mathcal{S}' \subseteq F^{-1}(F(\mathcal{S}'))$. On the other hand, for an object $X$ in $F^{-1}(F(\mathcal{S}'))$, there is an object $Y$ in $\mathcal{S}'$ satisfying $F(X) = F(Y)$ and therefore $X$ is in the $\nu$-orbit of $Y$. Since $\mathcal{S}'$ is Nakayama-stable, $X$ is also in $\mathcal{S}'$ and $F^{-1}(F(\mathcal{S}')) \subseteq \mathcal{S}'$. Therefore $F^{-1}(F(\mathcal{S}')) = \mathcal{S}'$.

For a Nakayama-stable orthogonal system $\mathcal{S}'$ in $A$-$\underline{\mathrm{mod}}$, using the formula $\coprod_{F(c)=F(b)} \underline{\mathrm{Hom}}_A(a,c) \cong \underline{\mathrm{Hom}}_B(F(a),F(b))$, we find that $F(\mathcal{S}')$ is an orthogonal system in $B$-$\underline{\mathrm{mod}}$. Since $F$ is a covering functor, $F(\mathcal{S}')$ is an sms of $B$-$\underline{\mathrm{mod}}$ if and only if $F^{-1}(F(\mathcal{S}')) = \mathcal{S}'$ is an sms of $A$-$\underline{\mathrm{mod}}$.

From the above discussion, we know that there is a bijection between the sms's of $A$-$\underline{\mathrm{mod}}$ and the sms's of $B$-$\underline{\mathrm{mod}}$ induced by $F$. ∎

By the above lemma, for a self-injective Nakayama algebra $A_n^\ell$, we know that $\mathcal{S}$ is an sms of $A_e^\ell$-$\underline{\mathrm{mod}}$ if and only if $F^{-1}(\mathcal{S})$ is an sms of $A_n^\ell$-$\underline{\mathrm{mod}}$. Since $e$ divides $\ell$, $A_e^\ell$ is a symmetric Nakayama algebra and therefore we have two types of sms's $\mathcal{L}_{p,k}$ and $\mathcal{S}_{p,k}$, where $p \in \mathcal{P}$, $1 \le k \le e$, and $\mathcal{P}$ is the set of non-crossing partitions of $\underline{e} = \{1,\dots,e\}$. Using the above covering functor we define two classes of objects in $A_n^\ell$-$\underline{\mathrm{mod}}$:

$$\mathcal{L}'_{p,k} := F^{-1}(\mathcal{L}_{p,k}), \qquad \mathcal{S}'_{p,k} := F^{-1}(\mathcal{S}_{p,k}).$$

Notice that the covering functor $F$ is induced from a covering of stable Auslander–Reiten quivers $\pi\colon {}_s\Gamma_{A_n^\ell} \to {}_s\Gamma_{A_e^\ell} \cong {}_s\Gamma_{A_n^\ell}/\langle\nu\rangle$ (where $\nu$ is the

Nakayama automorphism of ${}_s\Gamma_{A_n^\ell}$), therefore it is very easy to construct $\mathcal{L}'_{p,k}$ and $\mathcal{S}'_{p,k}$ from $\mathcal{L}_{p,k}$ and $\mathcal{S}_{p,k}$ in practice. We have the following theorem.

THEOREM 4.16. *Let $A_n^\ell$ be a self-injective Nakayama algebra and $\mathcal{P}$ the set of non-crossing partitions of $\underline{e}$, where $e = \gcd(n, \ell)$. Then we have the following:*

(a) *For any $p \in \mathcal{P}$ and any $1 \le k \le e$, $\mathcal{L}'_{p,k}$ and $\mathcal{S}'_{p,k}$ are sms's.*
(b) *All sms's of $A_n^\ell$ are of these forms.*
(c) *If $\ell/e \ge 2$, then for $p, p' \in \mathcal{P}$ and $1 \le k, k' \le e$ we have the following results:*
    (1) $\mathcal{L}'_{p,k} \ne \mathcal{S}'_{p',k'}$.
    (2) $\mathcal{L}'_{p,k} = \mathcal{L}'_{p',k'}$ *if and only if $p = p'$ and $p(k) = p(k')$.*
    (3) $\mathcal{S}'_{p,k} = \mathcal{S}'_{p',k'}$ *if and only if the following three conditions hold:* (i) $p = p'$; (ii) $k = k'$ *or* $\widehat{k} \cap \widehat{k'} = \emptyset$; (iii) $p_{k\vee k'} \in \mathcal{P}$, *where*

$$p_{k\vee k'}(i) = \begin{cases} p(k) \cup p(k'), & i \in p(k) \cup p(k'), \\ p(i), & \textit{otherwise}. \end{cases}$$

*Proof.* By Lemma 4.15, there is a covering functor $F\colon A_n^\ell\text{-}\underline{\mathrm{mod}} \to A_e^\ell\text{-}\underline{\mathrm{mod}}$.

(a) For any $p \in \mathcal{P}$ and any $1 \le k \le e$, we have $\mathcal{L}'_{p,k} = F^{-1}(\mathcal{L}_{p,k})$ and $\mathcal{S}'_{p,k} = F^{-1}(\mathcal{S}_{p,k})$. Since $\mathcal{L}_{p,k}$ and $\mathcal{S}_{p,k}$ are sms's of $A_e^\ell$-$\underline{\mathrm{mod}}$, by Lemma 4.15, $\mathcal{L}'_{p,k}$ and $\mathcal{S}'_{p,k}$ are sms's of $A_n^\ell$-$\underline{\mathrm{mod}}$.

(b) Take an sms $\mathcal{S}'$ of $A_n^\ell$-$\underline{\mathrm{mod}}$. By Lemma 4.15, $F(\mathcal{S}')$ is an sms of $A_e^\ell$-$\underline{\mathrm{mod}}$. By Theorem 4.10, $F(\mathcal{S}')$ is $\mathcal{L}_{p,k}$ or $\mathcal{S}_{p,k}$ for some $p \in \mathcal{P}$ and some $1 \le k \le e$. Moreover, by Lemma 4.15, $F^{-1}(F(\mathcal{S}')) = \mathcal{S}'$ and therefore $\mathcal{S}'$ is $\mathcal{L}'_{p,k}$ or $\mathcal{S}'_{p,k}$ for some $p \in \mathcal{P}$ and some $1 \le k \le e$.

(c) By Lemma 4.15, there is a bijection between the sms's of $A_n^\ell$-$\underline{\mathrm{mod}}$ and of $A_e^\ell$-$\underline{\mathrm{mod}}$ induced by $F$. By Theorem 4.10, conditions (1)–(3) are satisfied. ∎

EXAMPLE 4.17. We describe the sms's of the self-injective Nakayama algebra $A_4^6$. We know the sms's of $A_2^6$ from Example 4.12. Let $\mathcal{P} = \{p_1, p_2\}$ be the set of non-crossing partitions of $\underline{2}$, where $p_1 = \{\{1\}, \{2\}\}$, $p_2 = \{\{1, 2\}\}$. Then we can easily write down all sms's of $A_4^6$:

$$\mathcal{L}'_{p_1,1} = \left\{ \begin{matrix}1\\2\\3\\4\\1\end{matrix}, \begin{matrix}3\\4\\1\\2\\3\end{matrix}, 2, 4 \right\}, \quad \mathcal{S}'_{p_2,1} = \left\{ \begin{matrix}1\\2\end{matrix}, \begin{matrix}3\\4\end{matrix}, \begin{matrix}2\\3\\4\\1\\2\\3\end{matrix}, \begin{matrix}4\\1\\2\\3\\4\\1\end{matrix} \right\}, \quad \mathcal{L}'_{p_1,2} = \left\{ 1, 3, \begin{matrix}2\\3\\4\\1\\2\end{matrix}, \begin{matrix}4\\1\\2\\3\\4\end{matrix} \right\},$$

$$\mathcal{S}'_{p_2,2} = \left\{\begin{array}{llll} 1 & 3 & & \\ 2 & 4 & & \\ 3 & 1 & 2 & 4 \\ 4, & 2, & 3, & 1 \\ 1 & 3 & & \\ 2 & 4 & & \end{array}\right\}, \qquad \mathcal{L}'_{p_2,1} = \mathcal{L}'_{p_2,2} = \left\{\begin{array}{llll} 1 & 3 & 2 & 4 \\ 2 & 4 & 3 & 1 \\ 3 & 1 & 4 & 2 \\ 4, & 2, & 1, & 3 \\ 1 & 3 & 2 & 4 \\ 2 & 4 & 3 & 1 \end{array}\right\},$$

$$\mathcal{S}'_{p_1,1} = \mathcal{S}'_{p_1,2} = \{1, 3, 2, 4\}.$$

REMARK 4.18. In view of the observation in Remark 4.9, we will call $\mathcal{L}_{p,k}$ (or $\mathcal{L}'_{p,k}$) an sms *of long type* and $\mathcal{S}_{p,k}$ (or $\mathcal{S}'_{p,k}$) an sms *of short type.*

REMARK 4.19. Using some descriptions of non-crossing partitions from [11], Wenting Huang ([1]), an undergraduate student from BNU, devised an algorithm for constructing sms's over self-injective Nakayama algebras [8].

**5. Sms's of $A_n^\ell$ under the (co)syzygy functor.** This section is devoted to the behavior of sms's over $A_n^\ell$ under the (co)syzygy functor.

**5.1. Permutations over the set $\mathcal{P}$ of non-crossing partitions.** We fix some notations from previous sections. We denote by $\mathcal{P}$ the set of non-crossing partitions of $\underline{n} = \{1, \ldots, n\}$, and given $i \in \mathbb{Z}$ we denote by $\overline{i}$ the positive integer in $\underline{n}$ such that $i \equiv \overline{i} \bmod n$. For $p \in \mathcal{P}$, we denote by $p(i)$ the block which $i$ belongs to and we assume that $p(i) = \{i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}\}$ with $i, k_i, k_i^{(2)}, \ldots, k_i^{(s_i-1)}$ arranged anti-clockwise on the associated circle (cf. Corollary 4.4).

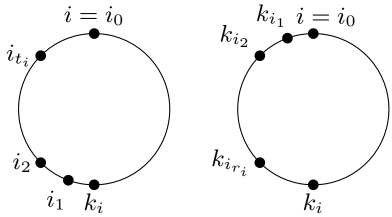
With any $p \in \mathcal{P}$ and $i \in \underline{n}$, we associate a subset $p'(i)$ of $\underline{n}$, where $p'(i) = \{i_{t_i}, \ldots, i_1, i\}$ is defined as follows: $i = i_0 = i_{t_i+1}$ and $i_m = \overline{k_{i_{m-1}} + 1}$ for each $1 \leq m \leq t_i + 1$. Suppose that $t$ is the least number satisfying $i_s = i_t$ for some $0 \leq s < t$. Then $i_t = i = i_0$, since otherwise, by the definitions of $i_s$ and $i_t$, $\overline{k_{i_{s-1}} + 1} = \overline{k_{i_{t-1}} + 1}$, $k_{i_{s-1}} = k_{i_{t-1}}$, and therefore $i_{s-1} = i_{t-1}$, a contradiction. Thus $p'(i)$ is well-defined. Moreover, with the above $p$ and $i$, we associate another subset $p''(k_i)$ of $\underline{n}$, where $p''(k_i) = \{k_i, k_{i_1}, \ldots, k_{i_{r_i}}\}$ is defined as follows: $i = i_0$, $k_i = k_{i_{r_i+1}}$ and $k_{i_n} = \overline{i_{n-1} - 1}$ for each $1 \leq n \leq r_i + 1$. Similarly, we can show that $p''(k_i)$ is well-defined.

LEMMA 5.1. *For any $p \in \mathcal{P}$, $p'$ and $p''$ define two partitions of $\underline{n}$.*

*Proof.* This is clear from the cyclic orderings in $p'(i)$ and $p''(k_i)$. ∎

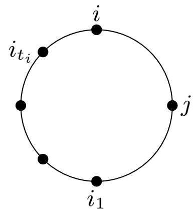
We illustrate the subsets $p'(i)$ and $p''(k_i)$ in the following two pictures:

([1]) Wenting Huang's email address: 877977235@qq.com.




LEMMA 5.2. *Let the partitions $p'$ and $p''$ be defined as above from the non-crossing partition $p$. Then $p'$ and $p''$ are non-crossing partitions.*

*Proof.* For any different blocks $p'(i)$ and $p'(j)$, we assume that $p'(i) = \{i_{t_i}, \dots, i_1, i\}$, where $i = i_0 = i_{t_i+1}$ and $i_m = \overline{k_{i_{m-1}} + 1}$ for each $1 \le m \le t_i + 1$, and $p'(j) = \{j_{t_j}, \dots, j_1, j\}$, where $j = j_0 = j_{t_j+1}$ and $j_m = \overline{k_{j_{m-1}} + 1}$ for each $1 \le m \le t_j + 1$. Without loss of generality, we assume that $j$ is a vertex on the arc $\widehat{ii_1}$, that is,




Since $p$ is a non-crossing partition, $k_j$ is a vertex on the arc $\widehat{ik_i}$. Therefore $j_1 = \overline{k_j + 1}$ is a vertex on the arc from vertex $i$ to vertex $i_1$. Similarly, any vertex in $p'(j)$ is on the arc from $i$ to $i_1$. Thus $p'$ is a non-crossing partition.

The proof for $p''$ is similar. ■

By the above lemma, we get two non-crossing partitions $p'$ and $p''$ from any non-crossing partition $p$ in $\mathcal{P}$. This suggests the following two self-maps of $\mathcal{P}$:

$$\mathcal{P} \xrightarrow{m_1} \mathcal{P}, \quad p \to p', \qquad \mathcal{P} \xrightarrow{m_2} \mathcal{P}, \quad p \to p''.$$

It is easy to check that $m_1 m_2 = \mathrm{id}$ and $m_2 m_1 = \mathrm{id}$, and therefore $m_1$ and $m_2$ are inverse bijections of $\mathcal{P}$.

EXAMPLE 5.3. Consider the non-crossing partition $p = \{\{1, 6, 4\}, \{2, 3\}, \{5\}\}$ of $\underline{6}$. By a direct computation,

$$p \xrightarrow{m_1} p' \xrightarrow{m_2} p \xrightarrow{m_2} p'' \xrightarrow{m_1} p,$$

where $p' = \{\{1\}, \{4, 2\}, \{3\}, \{6, 5\}\}$ and $p'' = \{\{1, 3\}, \{2\}, \{4, 5\}, \{6\}\}$.

**5.2. The behaviors of sms's under $\Omega$ and $\Omega^{-1}$.** Recall that for any indecomposable $A_n^{dn}$-module $M$, if $\mathrm{top}(M) \cong S_i$, $\mathrm{soc}(M) \cong S_j$ and the multiplicity of $S_i$ in $M$ is $k + 1$, then we denote $M$ by $M^i_{j,k}$. For any $A_n^{dn}$-module $M^i_{k_i,l_i}$, we have the following lemma about $\Omega(M^i_{k_i,l_i})$ and $\Omega^{-1}(M^i_{k_i,l_i})$, where $\Omega$, $\Omega^{-1}$ denote the syzygy and cosyzygy functors respectively.

Lemma 5.4. *Let $A_n^{dn}$ be a symmetric Nakayama algebra and $M^i_{k_i,l_i}$ an indecomposable $A_n^{dn}$-module. Then $\Omega(M^i_{k_i,l_i}) \cong M^{\overline{k_i+1}}_{i,d-l_i-1}$ and $\Omega^{-1}(M^i_{k_i,l_i}) \cong M^{k_i}_{\overline{i-1},d-l_i-1}$, where $\Omega$, $\Omega^{-1}$ are the syzygy and cosyzygy functors respectively.*

*Proof.* We only prove $\Omega(M^i_{k_i,l_i}) \cong M^{\overline{k_i+1}}_{i,d-l_i-1}$, since the other proof is dual. There is a short exact sequence

$$0 \to \Omega(M^i_{k_i,l_i}) \to P_i \to M^i_{k_i,l_i} \to 0,$$

where $P_i \to M^i_{k_i,l_i}$ is the projective cover of $M^i_{k_i,l_i}$.

We have $\Omega(M^i_{k_i,l_i}) \cong \mathrm{rad}^{nl_i+[k_i-i)+1}(P_i)$, where $[k_i-i)$ is the smallest nonnegative integer with $[k_i-i) = (k_i-i) \bmod n$. Therefore, $\mathrm{top}(\Omega(M^i_{k_i,l_i})) \cong \mathrm{top}(\mathrm{rad}^{nl_i+[k_i-i)+1}(P_i)) \cong S_{\overline{k_i+1}}$ and $\mathrm{soc}(\Omega(M^i_{k_i,l_i})) \cong \mathrm{soc}(P_i) \cong S_i$. Moreover, if $\overline{k_i+1} \neq \overline{i}$ (respectively $\overline{k_i+1} = \overline{i}$), then the multiplicity of $S_{\overline{k_i+1}}$ in $P_i$ is $d$ (respectively $d+1$) and the multiplicity of $S_{\overline{k_i+1}}$ in $M^i_{k_i,l_i}$ is $l_i$ (respectively $l_i+1$). Therefore the multiplicity of $S_{\overline{k_i+1}}$ in $\Omega(M^i_{k_i,l_i})$ is $d-l_i$ and $\Omega(M^i_{k_i,l_i}) \cong M^{\overline{k_i+1}}_{i,d-l_i-1}$. ∎

For $\mathcal{S}_{p,k}$, we define $\Omega(\mathcal{S}_{p,k}) = \{\Omega(M^i_{k_i,l_i}) \mid M^i_{k_i,l_i} \in \mathcal{S}_{p,k}\}$ and $\Omega^{-1}(\mathcal{S}_{p,k}) = \{\Omega^{-1}(M^i_{k_i,l_i}) \mid M^i_{k_i,l_i} \in \mathcal{S}_{p,k}\}$. Similarly, for $\mathcal{L}_{p,k}$, we can define $\Omega(\mathcal{L}_{p,k})$ and $\Omega^{-1}(\mathcal{L}_{p,k})$. From the above lemma and notations, we have the following theorem.

Theorem 5.5. *Let $A_n^{dn}$ be a symmetric Nakayama algebra and $\mathcal{P}$ the set of non-crossing partitions of the set $\underline{n}$. For $p \in \mathcal{P}$ and $1 \le i \le n$, let $\mathcal{L}_{p,i}$ and $\mathcal{S}_{p,i}$ be as in Definition 4.8. Moreover, let $m_1$ and $m_2$ be permutations of $\mathcal{P}$ defined as in Subsection 5.1, and let $\Omega$, $\Omega^{-1}$ denote the syzygy and cosyzygy functors respectively. Then we have the following:*

(1) $\Omega(\mathcal{S}_{p,i}) = \mathcal{L}_{p',i}$, *where* $p' = m_1(p)$.
(2) $\Omega^{-1}(\mathcal{L}_{p,i}) = \mathcal{S}_{p'',i}$, *where* $p'' = m_2(p)$.
(3) $\Omega^{-1}(\mathcal{S}_{p,i}) = \mathcal{L}_{p'',k_i}$, *where $k_i$ is defined as in Subsection* 5.1 *and* $p'' = m_2(p)$.
(4) $\Omega(\mathcal{L}_{p,i}) = \mathcal{S}_{p',i_1}$, *where $i_1 = \overline{k_i+1}$ is defined as in Subsection* 5.1 *and* $p' = m_1(p)$.

*Proof.* (1) Since $\Omega\colon A_n^{dn}\text{-}\underline{\mathrm{mod}} \to A_n^{dn}\text{-}\underline{\mathrm{mod}}$ is a stable equivalence, $\Omega(\mathcal{S}_{p,i})$ is also an sms. By Lemma 5.4, the non-crossing partition corresponding to $\Omega(\mathcal{S}_{p,i})$ is exactly $p' = m_1(p)$. For any vertex $j$, we denote by $p(j)$ the block of the non-crossing partition $p$ that $j$ belongs to, and let $p(j) = \{j, k_j, k_j^{(2)}, \ldots, k_j^{(s_j-1)}\}$ and $\widehat{j} = \{j, \overline{j+1}, \ldots, k_j\}$ be as before. Notice that when $j$ is an element in $p'(i)$ different from $i$, we have $\widehat{j} \cap p(i) = \emptyset$. Let $M^j_{k_j,l_j} \in \mathcal{S}_{p,i}$. Since $\mathrm{top}(M^j_{k_j,l_j}) \cong S_j$ and $\mathrm{soc}(M^j_{k_j,l_j}) \cong S_{k_j}$, $\mathrm{top}(\Omega(M^j_{k_j,l_j})) \cong S_{\overline{k_j+1}}$ by Lemma 5.4. By the definition of $\mathcal{S}_{p,i}$, we find that if $j$ is in $p'(i)$, then $l_j = 0$ and therefore the multiplicity of $S_{\overline{k_j+1}}$ in $\Omega(M^j_{k_j,l_j})$ is $d$. By Remark 4.9, we have $\Omega(\mathcal{S}_{p,i}) = \mathcal{L}_{p',i}$.

(2) Applying the functor $\Omega^{-1}$ to the equation in (1), we get the desired result.

(3) Since $\Omega^{-1}\colon A_n^{dn}\text{-}\underline{\mathrm{mod}} \to A_n^{dn}\text{-}\underline{\mathrm{mod}}$ is a stable equivalence, $\Omega^{-1}(\mathcal{S}_{p,i})$ is also an sms. By Lemma 5.4, the non-crossing partition corresponding to $\Omega^{-1}(\mathcal{S}_{p,i})$ is exactly $p'' = m_2(p)$. For any vertex $j$, we denote by $p(j)$ the block of $p$ that $j$ belongs to. Let $p(i) = \{i, k_i, k_i^{(2)}, \dots, k_i^{(s_i-1)}\}$ and $\widehat{j} = \{j, \overline{j+1}, \dots, k_j\}$ for any vertex $j$. Notice that when $k_j$ is an element in $p''(k_i)$ different from $k_i$, we have $\widehat{j} \cap p(i) = \emptyset$. Let $M^j_{k_j,l_j} \in \mathcal{S}_{p,i}$. Since $\mathrm{soc}(M^j_{k_j,l_j}) \cong S_{k_j}$, $\mathrm{top}(\Omega^{-1}(M^j_{k_j,l_j})) \cong S_{k_j}$ by Lemma 5.4. By the definition of $\mathcal{S}_{p,i}$, if $k_j$ is in $p''(k_i)$, then $l_j = 0$ and therefore the multiplicity of $S_{k_j}$ in $\Omega^{-1}(M^j_{k_j,l_j})$ is $d$. By Remark 4.9, we have $\Omega^{-1}(\mathcal{S}_{p,i}) = \mathcal{L}_{p'',k_i}$.

(4) Applying the functor $\Omega$ to the equation in (3), we get the desired result. ∎

Remark 5.6. Notice that for $p, p' \in \mathcal{P}$ and $1 \le k, k' \le n$, we find that $\mathcal{S}_{p,k} = \mathcal{S}_{p',k'}$ if and only if $\Omega(\mathcal{S}_{p,k}) = \Omega(\mathcal{S}_{p',k'})$, and by Theorem 5.5, if and only if $\mathcal{L}_{m_1(p),k} = \mathcal{L}_{m_1(p'),k'}$. By Theorem 4.10, for $A_n^{dn}$ and $d \ge 2$, $\mathcal{S}_{p,k} = \mathcal{S}_{p',k'}$ if and only if $p = p'$ and $m_1(p)(k) = m_1(p)(k')$.

Example 5.7. Consider the symmetric Nakayama algebra $A_2^6$ and the set $\mathcal{P}$ of non-crossing partitions of $\{1,2\}$; then $\mathcal{P} = \{p_1, p_2\}$ and $p_1 = \{\{1\},\{2\}\}$, $p_2 = \{\{1,2\}\}$.

By the definitions of $m_1$ and $m_2$, we have $m_1 = m_2\colon p_1 \mapsto p_2, p_2 \mapsto p_1$. For example,

$$\mathcal{S}_{p_2,1} = \left\{ \begin{matrix} 1 \\ 2 \end{matrix} , \begin{matrix} 2 \\ 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{matrix} \right\}, \qquad \mathcal{L}_{p_1,1} = \left\{ \begin{matrix} 1 \\ 2 \\ 1 \\ 2 \\ 1 \end{matrix} , 2 \right\}.$$

Obviously, $\Omega(\mathcal{S}_{p_2,1}) = \mathcal{L}_{p_1,1} = \Omega^{-1}(\mathcal{S}_{p_2,2}), \Omega^{-1}(\mathcal{L}_{p_1,1}) = \mathcal{S}_{p_2,1} = \Omega(\mathcal{L}_{p_1,2})$. Similarly,

$$\begin{aligned} \Omega(\mathcal{S}_{p_2,2}) = \mathcal{L}_{p_1,2} = \Omega^{-1}(\mathcal{S}_{p_2,1}), &\quad \Omega^{-1}(\mathcal{L}_{p_1,2}) = \mathcal{S}_{p_2,2} = \Omega(\mathcal{L}_{p_1,1}), \\ \Omega(\mathcal{S}_{p_1,1}) = \Omega(\mathcal{S}_{p_1,2}) &= \mathcal{L}_{p_2,2} = \mathcal{L}_{p_2,1} = \Omega^{-1}(\mathcal{S}_{p_1,1}), \\ \Omega^{-1}(\mathcal{L}_{p_2,1}) = \Omega^{-1}(\mathcal{L}_{p_2,2}) &= \mathcal{S}_{p_1,1} = \mathcal{S}_{p_1,2} = \Omega(\mathcal{L}_{p_2,1}). \end{aligned}$$

Similarly, for any self-injective Nakayama algebra $A_n^\ell$, and $\mathcal{L}'_{p,k}$ and $\mathcal{S}'_{p,k}$ over $A_n^\ell$, by Theorems 4.16 and 5.5 we have the following result.

THEOREM 5.8. *Let $A_n^\ell$ be a self-injective Nakayama algebra and $\mathcal{P}$ the set of non-crossing partitions of the set $\underline{e}$, where $e$ is the greatest common divisor of $n$ and $\ell$. For $p \in \mathcal{P}$ and $1 \leq i \leq e$, let $\mathcal{L}'_{p,i}$ and $\mathcal{S}'_{p,i}$ be as in Subsection 4.2. Moreover, let $m_1$ and $m_2$ be bijections of $\mathcal{P}$ as in Subsection 5.1, and let $\Omega$, $\Omega^{-1}$ denote the syzygy and cosyzygy functors respectively. Then we have the following:*

(1) $\Omega(\mathcal{S}'_{p,i}) = \mathcal{L}'_{p',i}$, *where* $p' = m_1(p)$.
(2) $\Omega^{-1}(\mathcal{L}'_{p,i}) = \mathcal{S}'_{p'',i}$, *where* $p'' = m_2(p)$.
(3) $\Omega^{-1}(\mathcal{S}'_{p,i}) = \mathcal{L}'_{p'',k_i}$, *where $k_i$ is defined as in Subsection* 5.1 *and* $p'' = m_2(p)$.
(4) $\Omega(\mathcal{L}'_{p,i}) = \mathcal{S}'_{p',i_1}$, *where $i_1 = \overline{k_i + 1}$ is defined as in Subsection* 5.1 *and* $p' = m_1(p)$.

EXAMPLE 5.9. Consider the self-injective Nakayama algebra $A_4^6$. From the last example, we have $m_1 = m_2 \colon p_1 \mapsto p_2$, $p_2 \mapsto p_1$, where $p_1 = \{\{1\}, \{2\}\}$ and $p_2 = \{\{1, 2\}\}$. Similarly,

$$\Omega(\mathcal{S}'_{p_2,1}) = \mathcal{L}'_{p_1,1} = \Omega^{-1}(\mathcal{S}'_{p_2,2}), \quad \Omega(\mathcal{S}'_{p_2,2}) = \mathcal{L}'_{p_1,2} = \Omega^{-1}(\mathcal{S}'_{p_2,1}),$$
$$\Omega^{-1}(\mathcal{L}'_{p_1,1}) = \mathcal{S}'_{p_2,1} = \Omega(\mathcal{L}'_{p_1,2}), \quad \Omega^{-1}(\mathcal{L}'_{p_1,2}) = \mathcal{S}'_{p_2,2} = \Omega(\mathcal{L}'_{p_1,1}),$$
$$\Omega(\mathcal{S}'_{p_1,1}) = \Omega(\mathcal{S}'_{p_1,2}) = \mathcal{L}'_{p_2,2} = \mathcal{L}'_{p_2,1} = \Omega^{-1}(\mathcal{S}'_{p_1,1}),$$
$$\Omega^{-1}(\mathcal{L}'_{p_2,1}) = \Omega^{-1}(\mathcal{L}'_{p_2,2}) = \mathcal{S}'_{p_1,1} = \mathcal{S}'_{p_1,2} = \Omega(\mathcal{L}'_{p_2,1}).$$

**5.3. The number of sms's of Brauer tree algebras.** Let $A_n^{dn}$ be a symmetric Nakayama algebra and $\mathcal{P}$ the set of non-crossing partitions of $\underline{n} = \{1, \ldots, n\}$, where $n, d$ are positive integers. The number of sms's in $\{\mathcal{S}_{p,k} \mid p \in \mathcal{P},\ k \in \underline{n}\}$ is equal to the number of sms's in $\{\mathcal{L}_{p,k} \mid p \in \mathcal{P},\ k \in \underline{n}\}$ by Theorem 5.5. Since the number of non-crossing partitions of $\underline{n}$ with $k$ blocks is the Narayana number $N(n,k)$ (see [10, Corollary 4.1]), where $N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1}$. By Theorem 4.10, if $d \geq 2$, then the number of sms's in $\{\mathcal{L}_{p,k} \mid p \in \mathcal{P},\ k \in \underline{n}\}$ is $\sum_{k=1}^{n} kN(n,k)$. Otherwise, the number of elements in $\{\mathcal{L}_{p,k} \mid p \in \mathcal{P},\ k \in \underline{n}\}$ is $\sum_{k=1}^{n} N(n,k) = C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the Catalan number and it is also the number of non-crossing partitions of $\underline{n}$.

Now we can easily calculate the number of sms's over any self-injective Nakayama algebra. Actually, it was already calculated by Riedtmann [13] and Chan [5] using their respective classifications.

PROPOSITION 5.10. *Let $A_n^\ell$ be a self-injective Nakayama algebra and $\mathcal{P}$ the set of non-crossing partitions of $\underline{e}$, where $e = \gcd(n, \ell)$. If $\ell = e$, then the number of sms's over $A_n^\ell$ is the Catalan number $C_e$. If $\ell > e$, then the number of sms's over $A_n^\ell$ is $(e+1)C_e$, where $C_e = \frac{1}{e+1}\binom{2e}{e}$.*

*Proof.* Since the number of sms's over $A_n^\ell$ is equal to the number of sms's over $A_e^\ell$, we just consider the symmetric Nakayama algebra $A_e^\ell$.

If $\ell = e$, then by Theorem 4.10 and Remark 4.11 the number of sms's over $A_e^\ell$ is equal to the number of non-crossing partitions of $\underline{e}$, that is, the Catalan number $C_e$.

If $\ell > e$, then by Theorems 4.10, 4.16 and the discussion above, the number of sms's over $A_e^\ell$ is $2\sum_{k=1}^{e} kN(e,k)$, where $N(e,k)$ is the Narayana number. Notice that $N(e,k) = N(e, e-k+1)$.

When $e$ is even, we have

$$\begin{aligned} 2\sum_{k=1}^{e} kN(e,k) &= 2\sum_{k=1}^{e/2}\{kN(e,k) + (e-k+1)N(e,e-k+1)\} \\ &= 2(e+1)\sum_{k=1}^{e/2} N(e,k) = (e+1)C_e. \end{aligned}$$

When $e$ is odd, we have

$$\begin{aligned} &2\sum_{k=1}^{e} kN(e,k) \\ &= 2\sum_{k=1}^{(e-1)/2}\{kN(e,k) + (e-k+1)N(e,e-k+1)\} + (e+1)N\left(e, \frac{e+1}{2}\right) \\ &= 2(e+1)\sum_{k=1}^{(e-1)/2} N(e,k) + (e+1)N\left(e, \frac{e+1}{2}\right) \\ &= (e+1)\left\{C_e - N\left(e, \frac{e+1}{2}\right)\right\} + (e+1)N\left(e, \frac{e+1}{2}\right) = (e+1)C_e. \end{aligned}$$

The result follows directly from the above. ∎

Corollary 5.11. *Let $B = B(T)$ be a Brauer tree algebra defined by a Brauer tree $T$ with $n$ edges such that the multiplicity of the exceptional vertex of $T$ is $m_0$ (for the definition of Brauer tree algebra, we refer to* [16, Section 2]*). Let $\mathcal{P}$ be the set of non-crossing partitions of $\underline{n}$. If $m_0 = 1$, then the number of sms's over $B$ is the Catalan number $C_n$. If $m_0 > 1$, then the number of sms's over $B$ is $(n+1)C_n$, where $C_n = \frac{1}{n+1}\binom{2n}{n}$.*

*Proof.* This is a direct consequence of the fact that the Brauer tree algebra $B$ is stably equivalent to the symmetric Nakayama algebra $A_n^{nm_0}$ and the fact that sms's are invariant under stable equivalence. ∎

Our short-type and long-type sms's correspond exactly to Chan's bottom-type and top-type configurations defined in [5]. This can be observed by Chan's cutting-off procedure on configurations and by the fact that the (co)syzygy functors interchange the types of sms's; we leave the details to the interested reader.

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